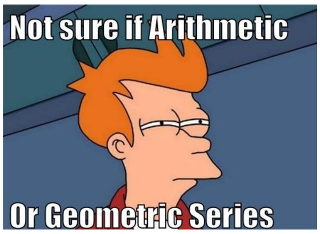
www.mymathscloud.com

Sequences & Series: Arithmetic & Geometric Solutions

(w 



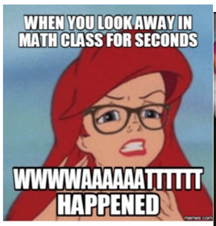


Table of Contents

1 Bronze

1.1 Without Formulae

1)

$$2n - 3$$

i.

n represents the term number

We want the 42nd term and hence we replace n with 42

$$2(42) - 3 = 81$$

ii. This is telling us that we want n (the term number) which gives a result of 1455

$$2n - 3 = 1455$$

We solve for n

$$2n - 3 = 1455$$
$$2n = 1458$$
$$n = 729$$

729^{th} term

2)

$$(n+3)(n-4)$$

i.

n represents the term number

We want the 20th term and hence we replace n with 20

$$(20+3)(20-4) = 368$$

ii. This is telling us that we want n (the term number) which gives a result of 78

$$(n+3)(n-4) = 78$$

We solve for n

$$(n+3)(n-4) = 78$$

Let's expand the brackets

$$n^2 - n - 12 = 78$$

We have a quadratic which means we need to get zero on one side and factorise or use the quadratic formula (this worksheet assumes you know how to solve quadratics and therefore does not go into detail on this).

$$n^2 - n - 90 = 0$$
$$(n+9)(n-10) = 0$$
$$n = -9 \text{ or } 10$$

But clearly n has to be a positive integer (since it represents term number), so $n = 10$.

10^{th} term

3)

We have $u_n = 3n + 2$. To find u_n, we simply plug in the required values of n.
We plug in $n = 1, 2, 3, 10$.

$$u_1 = 3(1) + 2 = 5$$
$$u_2 = 3(2) + 2 = 8$$
$$u_3 = 3(3) + 2 = 11$$
$$u_{10} = 3(10) + 2 = 32$$

4)

We are given that $u_n = 24$, and $u_n = 2n - 4$. So we can form an equation by setting them equal

$$24 = 2n - 4$$

Now we solve for n

$$28 = 2n$$
$$n = 14$$

5)

i.

We are given the n^{th} term as $u_n = 3 + 4n$

In order to find the terms asked for we replace every n we see with the values of the term number

first term= $u_1 = 3 + 4(1) = 7$

second term $= u_2 = 3 + 4(2) = 11$

third term $= u_3 = 3 + 4(3) = 15$

ii.

$u_n = 24$

We are given that $u_n = 3 + 4n$

So, we can form an equation

$$3 + 4n = 27$$
$$4n = 24$$
$$n = 6$$

iii.

Our sequence from part i. looks like 7, 11, 15, ...

We can see that we add 4 each time so we can find more of the sequence

$$7, 11, 15, 19, 23, 27, \ldots$$

$S_5 =$ sum of the first 5 terms $= 7 + 11 + 15 + 19 + 23 = 75$

6)

Remember, S_5 is the sum of first 5 terms. $S_5 = u_1 + u_2 + u_3 + u_4 + u_5$.

There are multiple ways of solving this problem

Method 1.

We use the formula $S_n = \frac{n}{2}[a + l]$, where a is the first term, l is the last ter

We work out the first term $a = u_1 = 2(1) + 5 = 7$

We work out the last term $l = u_5 = 2(5) + 5 = 15$

$$S_5 = \frac{5}{2}(7 + 15) = 55$$

Method 2.

We use the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$, where a is the first term, d is the common difference.

We work out the first term $a = u_1 = 2(1) + 5 = 7$.

Common difference $d = u_2 - u_1 = 2(2) + 5 - 7 = 2$.

In fact, d is always the coefficient of n in an arithmetic sequence, can you see why?

$$S_5 = \frac{5}{2}(2(7) + (5 - 1)2) = 55$$

Method 3.

Work out u_1 to u_5 individually and add them up (not a good method when we have a number much bigger than 5).

$$u_1 = 2(1) + 5 = 7$$
$$u_2 = 2(2) + 5 = 9$$
$$u_3 = 2(3) + 5 = 11$$
$$u_4 = 2(4) + 5 = 13$$
$$u_5 = 2(5) + 5 = 15$$

$$S_5 = 7 + 9 + 11 + 13 + 15 = 55$$

7)

What do we know about an arithmetic sequence? If we subtract the successive terms, they must give the same answer.

$$2p - (12 - p) = (4p - 5) - 2p$$
$$2p - 12 + p = 4p - 5 - 2p$$
$$p = 7$$

Put this back into sequence

$$12 - 7, 2(7), 4(7) - 5$$
$$5,14,23$$
$$a = 5, d = 14 - 5 = 9$$

Note: we could have also done $12 - p - 2p = 2p - (4p - 5)$ etc.

8)

We approach it in a similar manner to the previous question. We know the quotient of successive terms are equal.

$$\frac{a}{a+14} = \frac{a+14}{9a}$$

$$9a^2 = (a + 14)^2$$

$$9a^2 = a^2 + 28a + 196$$

$$8a^2 - 28a - 196 = 0$$

$$2a^2 - 7a - 49 = 0$$

$$(2a + 7)(a - 7) = 0$$

So, we actually have two solutions, $a = 7 \text{ or } -\frac{7}{2}$.

9)

i. Plug in $x = 5$ to $x - 3, x + 1, 2x + 8$:

$$5 - 3, 5 + 1, 2(5) + 8$$
$$2, 6, 18$$

ii.

$$r = \frac{u_2}{u_1} = \frac{6}{2} = 3$$

iii. If the sequence is geometric, then the ratio of successive terms must be the same, so we can build the following equation:

$$\frac{x-3}{x+1} = \frac{x+1}{2x+8}$$

Now we solve by cross multiplying

$$(2x+8)(x-3) = (x+1)^2$$

Expand the brackets

$$2x^2 + 2x - 24 = x^2 + 2x + 1$$
$$x^2 - 25 = 0$$
$$x = \pm 5$$

The other value of x for which the sequence is geometric is -5.

iv. We first plug in $x = -5$ into $x - 3, x + 1$ to find the first two terms.

$$(-5) - 3, (-5) + 1$$
$$-8, -4$$
$$r = \frac{u_2}{u_1} = \frac{-4}{-8} = \frac{1}{2}$$

1.2 Using Formulae

10)

4 ,9, 14 ,19,…

first term $= a = 4$

common difference $= d = 9 - 4 = 5$

The formula for $u_n = a + (n-1)d$

Let's replace a and d in the formula

$$4 + (n-1)5$$
$$4 + 5n - 5$$
$$5n - 1$$

11)

i.

The common difference $d = u_2 - u_1 = 40 - 36 = 4$

ii.

We use the formula $u_n = a + (n-1)d$. a is the first term, which is 36.

$$u_n = 36 + (n-1)4$$
$$= 36 + 4n - 4$$
$$= 32 + 4n$$

iii.

We use the formula $S_n = \frac{n}{2}[2a + (n-1)d]$. We already have a, d from the previous part.

$$S_n = \frac{n}{2}(2(36) + (n-1)(4))$$
$$= \frac{n}{2}(72 + 4n - 4)$$
$$= \frac{n}{2}(68 + 4n)$$
$$= n(34 + 2n)$$
$$= 2n^2 + 34n$$

iv.

$$s_{14} = 2(14)^2 + 34(14) = 868$$

12)

i.

Let's work out the n^{th} term so we can substitute 101 in.

$$u_n = a + (n-1)d$$

The first term $= a = 2$

The common difference $= d = 5 - 2 = 3$

$$u_n = 2 + (n-1)(3)$$
$$= 2 + 3n - 3$$
$$= 3n - 1$$

Now, $u_{101} = 3(101) - 1 = 302$.

ii.

We have the equation $152 = u_n = 3n - 1$, we solve for n.

$$152 = 3n - 1$$

$$153 = 3n$$
$$n = 51$$

13)

i.

d is the common difference, $d = 8 - 5 = 3$

ii.

We first need to work out the n^{th} term.

$$u_n = a + (n-1)d$$

We already know $d = 3$, a is the first term, which is 5.

$$u_n = 5 + (n-1)(3)$$
$$u_n = 5 + 3n - 3$$
$$u_n = 2 + 3n$$

Now, plug in $n = 100$, $u_{100} = 2 + 3(100) = 302$.

iii.

We use the formula $S_n = \frac{n}{2}[a + l]$, where l is the last term. The reason we use this formula is because we already know the last term $l = u_{100} = 302$.

So, $s_{100} = \frac{100}{2}(5 + 302) = 15350$.

14)

i.

The common difference $d = 4 - 2 = 2$. So, the next term is $6 + 2 = 8$.

$$s_4 = u_1 + u_2 + u_3 + u_4$$
$$s_4 = 2 + 4 + 6 + 8 = 20$$

ii.

The method above is obviously not practical for finding s_{100}. We use the formula $s_n = \frac{n}{2}[2a + (n-1)d]$ instead.

We have the first term $a = 2$.
The common difference $d = 2$.
We also have $n = 100$.

$$s_{100} = \frac{100}{2}\left(2(2) + (100-1)(2)\right)$$
$$= 50(4 + 2(100) - 2)$$
$$= 50(202)$$
$$= 10100$$

15)

i.

$$d = u_2 - u_1 = 8 - 2 = 6$$

ii.

Using the formula $u_n = a + (n-1)d$
We know a is the first term, which is 2.

$$u_{20} = 2 + (20-1)(6)$$
$$= 2 + 114 = 116$$

iii.

We use the formula $S_n = \frac{n}{2}[a + l]$, we already have $a = 2$, and we worked out $l = u_{20} = 116$ from the previous part.

$$s_{20} = \frac{20}{2}(2 + 116) = 1180$$

16)

i.

We can use the formula $u_n = ar^{n-1}$.
We know the first term $a = 12$.
We can work out the common ratio $r = \frac{-6}{12} = -\frac{1}{2}$.

We find the tenth term $= u_{10} = 12\left(-\frac{1}{2}\right)^{10-1} = \frac{12}{-2^9} = -\frac{3}{2^7} = -\frac{3}{128}$

ii.

We use the formula $S_\infty = \frac{a}{1-r}$. We need to check that $-1 < r = -\frac{1}{2} < 1$, before we can use this formula, which is indeed the case.

$$S_\infty = \frac{12}{1-\left(-\frac{1}{2}\right)} = \frac{12}{\frac{3}{2}} = 8$$

17)

i.

We can use the formula $u_n = a + (n-1)d$ to find d, since we already know $a = u_1 = 2$.
If we plug in $n = 1$, then d would disappear, so we need to use $n = 20$.

$$u_{20} = 78 = 2 + (20-1)d$$
$$76 = 19d$$
$$d = 4$$

ii.

We now solve for n using the same formula.

$$u_n = 3710 = 2 + (n-1)(4)$$
$$3710 = 2 + 4n - 4$$
$$3712 = 4n$$
$$n = 928$$

18) In an arithmetic sequence the first term is 5 and the fourth term is 40, find the second term

We can use the formula $u_n = a + (n-1)d$.

We already have $a = u_1 = 5$.
We can solve for d using our knowledge of u_4.

$$u_4 = 40 = 5 + (4-1)d$$
$$35 = 3d$$
$$d = 5$$

Now, we plug these values back in.

$$u_2 = 5 + (2-1)(5) = 10$$

19)

i. We can use the formula $s_n = \frac{n}{2}[2a + (n-1)d]$.
We already know the first term $a = -7$.

$$s_{20} = 620 = \frac{20}{2}(2(-7) + (20-1)d)$$
$$620 = 10(-14 + 19d)$$
$$620 = -140 + 190d$$
$$760 = 190d$$
$$d = 4$$

The common difference is 4.

ii. We use the formula $u_n = a + (n-1)d$

$$u_{78} = -7 + (78-1)(4) = 301$$

2 Silver

2.1 Without Formulae

20)

Way 1:
Let's substitute $n = 1, 2, 3$ into s_n to find the first three terms of the series.

$$s_1 = 2(1)^2 - 1 = 1$$
$$s_2 = 2(2)^2 - 2 = 6$$
$$s_3 = 2(3)^2 - 3 = 15$$

We can find the corresponding u_n using the formula $u_n = s_n - s_{n-1}$, and that $u_1 = s_1$:

$$u_1 = s_1 = 1$$
$$u_2 = s_2 - s_1 = 6 - 1 = 5$$
$$u_3 = s_3 - s_2 = 15 - 6 = 9$$

To find the n^{th} term of the sequence, we need to find a, d.

$$a = u_1 = 1$$
$$d = u_2 - u_1 = 5 - 1 = 4$$

$$u_n = a + (n-1)d$$
$$u_n = 1 + (n-1)4$$
$$u_n = 1 + 4n - 4$$
$$u_n = 4n - 3$$

Way 2:

$$u_n = s_n - s_{n-1}$$
$$= 2n^2 - n - \left(2(n-1)^2 - (n-1)\right)$$
$$= 2n^2 - n - [2(n-1)(n-1) - n + 1]$$
$$= 2n^2 - n - [2(n^2 - 2n + 1) - n + 1]$$
$$= 2n^2 - n - [2(n^2 - 2n + 1) - n + 1]$$
$$= 2n^2 - n - [2n^2 - 4n + 2 - n + 1]$$
$$= 2n^2 - n - [2n^2 - 5n + 3]$$
$$= 2n^2 - n - 2n^2 + 5n - 3$$
$$= 4n - 3$$

21)

The first term, u_1 is equal to s_1, because there is only one term to sum.

$$u_1 = s_1 = 32(1) - 1^2 = 31$$

To find the common difference, we can find the second term u_2 and use our formula $d = u_2 - u_1$

$$u_2 = s_2 - s_1 = 32(2) - 2^2 - 31 = 29$$

$$d = u_2 - u_1 = 29 - 31 = -2$$

22)

First, we need to find u_n. Recall that $u_n = s_n - s_{n-1}$.

$$u_n = s_n - s_{n-1}$$
$$u_n = 4n^2 - 2n - \left(4(n-1)^2 - 2(n-1)\right)$$
$$u_n = 4n^2 - 2n - 4(n-1)^2 + 2(n-1)$$
$$u_n = 4n^2 - 2n - 4n^2 + 8n - 4 + 2n - 2$$
$$u_n = 8n - 6$$

So, we have

$$u_2 = 8(2) - 6 = 10$$
$$u_m = 8m - 6$$
$$u_{32} = 8(32) - 6 = 250$$

To form a geometric sequence, they must have a common ratio.

$$\frac{u_2}{u_m} = \frac{u_m}{u_{32}}$$
$$\frac{10}{8m-6} = \frac{8m-6}{250}$$
$$2500 = (8m-6)^2$$
$$2500 = 64m^2 - 96m + 36$$
$$64m^2 - 96m - 2464 = 0$$
$$2m^2 - 3m - 77 = 0$$
$$(2m+11)(m-7) = 0$$

The existence of term u_m implies that $m > 0$. So, $m = 7$.

2.2 Using Formulae

2.2.1 Finding n

23)

We have the first term $a = 5$; the common ratio $r = \frac{u_2}{u_1} = \frac{10}{5} = 2$.

So, the formula for the n^{th} term is $u_n = ar^{n-1} = 5(2)^{n-1}$.

To find the number of terms, we just need to find out which term is the last term, 1280. We can just use our n^{th} term formula and work backwards.

$$1280 = u_n = 5(2)^{n-1}$$
$$256 = (2)^{n-1}$$
$$8 = n - 1$$
$$n = 9$$

There are 9 terms in the sequence.

24)

$$a = \text{first term} = 3$$
$$d = \text{common difference} = 9 - 3 = 6$$

We know the final term is 1353. We need to know what term number this corresponds to.
We say $u_n = 1353$ and if we can find n then we know which term it is. We have a formula for u_n which we can use to then work backwards and solve for n.

$$u_n = 1353$$

Let's replace u_n with its formula

$$a + (n-1)d = 1353$$

Now let's replace a and d

$$3 + (n-1)(6) = 1353$$

$$3+6n-6=1353$$
$$6n-3=1353$$
$$6n=1356$$
$$n=226$$

Now that we know we have 226 terms we can find the sum of the sequence using the formulas for s_n where $n=226$

$$S_{226}=\frac{226}{2}[2(3)+(226-1)6]=113[6+225(6)]=153228$$

25)

The multiples of 3 over 100 are 102, 105, 107, …

This is an arithmetic sequence with $a=102$, $d=105-102=3$. So, the formula is

$$u_n=102+(n-1)3=99+3n$$

We can use the formula $s_n=\frac{n}{2}(a+l)$, where l is the last term. Clearly, l has to be the biggest multiple of 3 less than 1000, which is 999. Now, we need to work out which term 999 is (so we can plug in n). We equate $999=u_n=100+3n$

$$999=99+3n$$
$$900=3n$$
$$n=300$$

999 is the $300th$ term in this sequence.

So, the sum of all multiples of 3 between 100 and 1000 is s_{300}.

$$s_n=\frac{n}{2}(a+l)$$
$$s_{300}=\frac{300}{2}(102+999)$$
$$s_{300}=\frac{300}{2}(1101)=165150.$$

2.2.2 Simultaneous Equations

26)

Here we don't know a and d, so we will need to work backwards to find them by using the formula

The fourth term of an arithmetic sequence is 3 tells us that $u_4=3$
The sum of the first 6 terms of an arithmetic sequence tells us that $s_6=27$

Let's use the u_n and s_n formulae

$u_4=3$	$s_6=27$
The formula tells us that $a+3d=3$	The formula tells us that $\frac{6}{2}[2a+5(d)]=27$

So, we have two equations:

$$a+3d=3 \quad (1)$$
$$3[2a+5(d)]=27 \quad (2)$$

Simplifying both we get

$$a+3d=3$$
$$2a+5d=9$$

Solve simultaneously

$$a=12, d=-3$$

27)

Here we don't know a and d, so we will need to work backwards to find them by using the formula

The fortieth term of an arithmetic sequence is 106 tells us that $u_{40} = 106$
The sum of the first 40 terms of an arithmetic sequence tells us that $S_{40} = 1900$

Let's use the u_n and S_n formulae

$u_{40} = 106$	$S_{40} = 1900$
The formula tells us that $a + 39d = 106$	The formula tells us that $\frac{40}{2}[2a + 39(d)] = 1900$

So, we have two equations:

$a + 39d = 106$ ①
$20[2a + 39(d)] = 1900$ ②

Simplifying both we get

$a + 39d = 106$
$2a + 39d = 950$

Solve simultaneously

$a = 12, d = -3$

28)

Here we don't know a and r, so we will need to work backwards to find them by using the formula.

The fourth term of a geometric sequence is 10 tells us that $u_4 = 10$
The seventh term of a geometric sequence is 80 tells us that $u_7 = 80$

Let's use the u_n formulae

$u_4 = 10$	$u_7 = 80$
The formula tells us that $ar^3 = 10$	The formula tells us that $ar^6 = 80$

So, we have two equations:

$ar^3 = 10$ ①
$ar^6 = 80$ ②

Solve simultaneously

Way 1: Divide Successive Terms

$$\frac{ar^6}{ar^3} = \frac{80}{10}$$
$$r^3 = 8$$
$$r = 2$$

$ar^3 = 10$ so $a(8) = 10$ so $a = 1.25$

$$S_{20} = \frac{a(1 - r^n)}{1 - r}$$
$$= \frac{1.25(1 - 2^{20})}{1 - 2}$$
$$= 1310718.75$$

Way 2: Use Substitution

$ar^3 = 10$ ①
$$r^3 = \frac{10}{a}$$

$ar^6 = 80$ ②
$$a(r^3)^2 = 80$$
$$a\left(\frac{10}{a}\right)^2 = 80$$
$$\frac{100}{a} = 80$$
$$80a = 100$$
$$a = 1.25$$

$$r^3 = \frac{10}{1.25}$$
$$r^3 = 8$$
$$r = 2$$

29)

Here we don't know a and r, so we will need to work backwards to find them by using the formula

The sum to infinity of a geometric sequence is 16 tells us that $s_\infty = 16$
The sum of the first four terms is 1580 tells us that $s_4 = 15$

Let's use the s_∞ and s_n formulae

$s_\infty = 16$	$s_4 = 15$
The formula tells us that $\frac{a}{1-r} = 16$	The formula tells us that $\frac{a(1-r^4)}{1-r} = 15$

So, we have two equations:

$$\frac{a}{1-r} = 16 \; ①$$

$$\frac{a(1-r^4)}{1-r} = 15 \; ②$$

Solve simultaneously

Way 1: Divide Successive Terms	**Way 2: substitution re-arrange for $\frac{a}{1-r}$**	**Way 3: substitution re-arrange for a**	**Way 4: substitution re-arrange for $1-r$**
$\frac{a(1-r^4)}{1-r} \div \frac{a}{1-r} = \frac{15}{16}$ $\frac{a(1-r^4)}{1-r} \times \frac{1-r}{a} = \frac{15}{16}$ Cross cancel $1-r^4 = \frac{15}{16}$ $r^4 = \frac{1}{16}$ $r = \pm\frac{1}{2}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r} = 16$ $\frac{a}{1-\frac{1}{2}} = 16 \Rightarrow a = 8$ $\frac{a}{1+\frac{1}{2}} = 16 \Rightarrow a = 24$ So $a = 24, r = -\frac{1}{2}$ and $a = 8, r = \frac{1}{2}$	From ②, we have $\frac{a}{1-r}(1-r^4) = 15$ But from ①, we know $\frac{a}{1-r} = 16$ We can substitute this into ②. $16(1-r^4) = 15$ $1-r^4 = \frac{15}{16}$ $r^4 = \frac{1}{16}$ $r = \pm\frac{1}{2}$ Sub back into either equation to find a (choose less complicated one). $\frac{a}{1-r} = 16$ $\frac{a}{1-\frac{1}{2}} = 16 \Rightarrow a = 8$ $\frac{a}{1+\frac{1}{2}} = 16 \Rightarrow a = 24$ So $a = 24, r = -\frac{1}{2}$ and $a = 8, r = \frac{1}{2}$	We can re-arrange ① for a to get $a = 16(1-r)$. Substitute this into ②. $\frac{16(1-r)(1-r^4)}{1-r} = 15$ Cancel $(1-r)$. $16(1-r^4) = 15$ $1-r^4 = \frac{15}{16}$ $r^4 = \frac{1}{16}$ $r = \pm\frac{1}{2}$ Sub back into either equation to find a (choose less complicated one). $\frac{a}{1-r} = 16$ $\frac{a}{1-\frac{1}{2}} = 16 \Rightarrow a = 8$ $\frac{a}{1+\frac{1}{2}} = 16 \Rightarrow a = 24$ So $a = 24, r = -\frac{1}{2}$ and $a = 8, r = \frac{1}{2}$	We can re-arrange ① for a to get $1-r = \frac{a}{16}$. Substitute this into ②. $\frac{a(1-r^4)}{\frac{a}{16}} = 15$ $16(1-r^4) = 15$ $1-r^4 = \frac{15}{16}$ $r^4 = \frac{1}{16}$ $r = \pm\frac{1}{2}$ Sub back into either equation to find a (choose less complicated one). $\frac{a}{1-r} = 16$ $\frac{a}{1-\frac{1}{2}} = 16 \Rightarrow a = 8$ $\frac{a}{1+\frac{1}{2}} = 16 \Rightarrow a = 24$ So $a = 24, r = -\frac{1}{2}$ and $a = 8, r = \frac{1}{2}$

30)

Here we don't know a and d, so we will need to work backwards to find them by using the formula.

The sum of the first sixteen term the arithmetic sequence is 212 tells us that $s_{16} = 212$
The fifth term the arithmetic sequence is 8 tells us that $u_5 = 8$

Let's use the u_n and the s_n formulae

$s_{16} = 212$	$u_5 = 8$
The formula tells us that $\frac{16}{2}(a + 15d) = 8a + 120d = 212$	The formula tells us that $a + 4d = 8$

So, we have two equations:

$$8a + 120d = 212 \quad (1)$$
$$a + 4d = 8 \quad (2)$$

Solve simultaneously. Times (2) by 8 and subtract from (1).

$$88d = 148$$
$$d = \frac{37}{22}$$
$$a = \frac{14}{11}$$

31) The sum of the first three terms of a geometric sequence is 62.755, and the sum of the infinite sequence is 440. Find the common ratio.

Here we don't know a and r, so we will need to work backwards to find them by using the formula

The sum to infinity of a geometric sequence is 440 tells us that $s_\infty = 440$
The sum of the first three terms is 62.755 tells us that $s_3 = 62.755$

Let's use the s_∞ and s_n formulae

$s_\infty = 440$	$s_3 = 62.755$
The formula tells us that $\frac{a}{1-r} = 440$	The formula tells us that $\frac{a(1-r^3)}{1-r} = 62.755$

So, we have two equations:

$$\frac{a}{1-r} = 440 \quad (1)$$

$$\frac{a(1-r^3)}{1-r} = 62.755 \quad (2)$$

Solve simultaneously

Way 1: Divide Successive Terms	**Way 2: substitution re-arrange for $\frac{a}{1-r}$**	**Way 3: substitution re-arrange for a**	**Way 4: substitution re-arrange for $1-r$**
$\frac{a(1-r^3)}{1-r} \div \frac{a}{1-r} = \frac{62.755}{440}$ $\frac{a(1-r^3)}{1-r} \times \frac{1-r}{a} = \frac{62.755}{440}$ Cross cancel $1 - r^3 = \frac{62.755}{440}$ $r^3 = \frac{6859}{8000}$	From (2), we have $\frac{a}{1-r}(1-r^3) = 62.755$ But from (1), we know $\frac{a}{1-r} = 440$	We can re-arrange (1) for a to get $a = 440(1-r)$. Substitute this into (2). $\frac{440(1-r)(1-r^4)}{1-r} = 62.755$	We can re-arrange (1) for a to get $1 - r = \frac{a}{440}$. Substitute this into (2). $\frac{a(1-r^4)}{\frac{a}{440}} = 62.755$ $440(1-r^3) = 62.755$

	$r=\sqrt[3]{\frac{6859}{8000}}=\frac{19}{20}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r}=440$ $\frac{a}{1-\frac{19}{20}}=440 \Rightarrow a=440\left(\frac{1}{20}\right)=22$ So $a=22, r=19/20$.	We can substitute this into ②. $440(1-r^3)=62.755$ $1-r^3=\frac{62.755}{440}$ $r^3=\frac{6859}{8000}$ $r=\sqrt[3]{\frac{6859}{8000}}=\frac{19}{20}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r}=440$ $\frac{a}{1-\frac{19}{20}}=440 \Rightarrow a=$ $440\left(\frac{1}{20}\right)=22$ So $a=22, r=19/20$.	Cancel $(1-r)$. $440(1-r^3)=62.755$ $1-r^3=\frac{62.755}{440}$ $r^3=\frac{6859}{8000}$ $r=\sqrt[3]{\frac{6859}{8000}}=\frac{19}{20}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r}=440$ $\frac{a}{1-\frac{19}{20}}=440 \Rightarrow a=$ $440\left(\frac{1}{20}\right)=22$ So $a=22, r=19/20$.	$1-r^3=\frac{62.755}{440}$ $r^3=\frac{6859}{8000}$ $r=\sqrt[3]{\frac{6859}{8000}}=\frac{19}{20}$ sub back into either equation to find a (choose less complicated one) $\frac{a}{1-r}=440$ $\frac{a}{1-\frac{19}{20}}=440 \Rightarrow a=$ $440\left(\frac{1}{20}\right)=22$ So $a=22, r=19/20$.

32)

Here we don't know a and r, so we will need to work backwards to find them by using the formula.

The sum of first two terms of a geometric sequence is 6 tells us that $S_2=6$
The sum of first four terms of a geometric sequence is 30 tells us that $S_4=30$

Let's use the u_n formulae

$S_2=6$	$S_4=30$
The formula tells us that $\frac{a(1-r^2)}{1-r}=6$	The formula tells us that $\frac{a(1-r^4)}{1-r}=30$

So, we have two equations:

$$\frac{a(1-r^2)}{1-r}=6 \text{ ①}$$

$$\frac{a(1-r^4)}{1-r}=30 \text{ ②}$$

Solve simultaneously

Way 1: Divide Successive Terms	**Way 2: Use Substitution**
$\frac{a(1-r^2)}{1-r}\div\frac{a(1-r^4)}{1-r}=\frac{6}{30}$ $\frac{a(1-r^2)}{1-r}\times\frac{1-r}{a(1-r^4)}=\frac{1}{5}$ Cross cancelling, we get $\frac{1-r^2}{1-r^4}=\frac{1}{5}$ We can use the difference of two squares to factorise the denominator. $\frac{(1-r^2)}{(1-r^2)(1+r^2)}=\frac{1}{5}$ $\frac{1}{1+r^2}=\frac{1}{5}$ $1+r^2=5$	$\frac{a(1-r^2)}{1-r}=6$ ① $a=\frac{6(1-r)}{1-r^2}$ $\frac{a(1-r^4)}{1-r}=30$ ② $\frac{\frac{6(1-r)}{1-r^2}(1-r^4)}{1-r}=30$ $\frac{6(1-r)(1-r^4)}{(1-r^2)(1-r)}=30$ $\frac{6(1-r^4)}{1-r^2}=30$ $\frac{1-r^4}{1-r^2}=5$ We can use the difference of two squares to factorise the numerator.

	$r = \pm 2$ Sub it back into ① (the slightly easier option) we get $\frac{a(1-2^2)}{1-2} = 6$ $a\left(\frac{-3}{-1}\right) = 6$ $a = 2$ And $\frac{a(1-(-2)^2)}{1-(-2)} = 6$ $a\left(\frac{-3}{3}\right) = 6$ $a = -6$ So, we have $r = 2, a = 2$ and $r = -2, a = -6$	$\frac{(1-r^2)(1+r^2)}{1-r^2} = 5$ $1 + r^2 = 5$ $r = \pm 2$ Sub it back into ① (the slightly easier option) we get $\frac{a(1-2^2)}{1-2} = 6$ $a\left(\frac{-3}{-1}\right) = 6$ $a = 2$ And $\frac{a(1-(-2)^2)}{1-(-2)} = 6$ $a\left(\frac{-3}{3}\right) = 6$ $a = -6$ So, we have $r = 2, a = 2$ and $r = -2, a = -6$	

2.2.3 With Inequalities

33)

Hint: we need to solve $u_n > 0$ for n since the greatest possible sum of when we add the positive terms. Adding the terms once the sequence becomes negative will only decrease the sum and hence won't give us the greatest possible sum.

$$a = 85, d = -7$$

Here we have a **negative common difference** so greatest possible sum will be before we get any negative terms i.e. we only want to add positive terms so $u_n > 0$

$$a + (n-1)d > 0$$
$$85 - 7n + 7 > 0$$
$$92 - 7n > 0$$
$$-7n > -92$$
$$n < 13.14$$

First term that satisfies this is $n = 13$

So 13 is the most number of terms we can have before the terms become negative and hence lessen the sum

So the greatest possible sum is the sum of the first 13 terms

$$S_{13} = \frac{12}{2}[2(85) + 12(-7)] = 559$$

34)

i.

We have $u_3 = 1407, u_{10} = 1183$.

We use the formula $u_n = a + (n-1)d$

$$1407 = a + 2d \text{①}$$
$$1183 = a + 9d \text{②}$$

Let's solve these equations simultaneously. Subtracting ① from ②,

$$-224 = 7d$$
$$d = -32$$

Substitute $d = -32$ back into ①.

$$1407 = a - 2(32)$$
$$a = 1471$$

ii.

The common difference is negative, so each successive term will decrease. We want to find the biggest n such that $u_n > 0$.

$$u_n = 1471 - 32(n-1) > 0$$
$$1471 - 32n + 32 > 0$$
$$1503 > 32n$$
$$46.96875 > n$$

The greatest n satisfying this constraint is $n = 46$, so there are 46 positive terms.

35)

We have $a = 1000, d = -6$. The sum of first n terms is negative. So,

$$s_n < 0$$

Using the formula for s_n

$$\frac{n}{2}\left(2(1000) + (n-1)(-6)\right) < 0$$
$$\frac{n}{2}(2000 - 6n + 6) < 0$$
$$n(1003 - 3n) < 0$$
$$n < 0 \text{ } or \text{ } n > \frac{1003}{3}$$

Since n has to be a positive integer, $n > \frac{1003}{3} \approx 334.333$.

The least n satisfying this inequality is 335.

36)

i.

We have $u_5 = 16, u_{10} = 30$.

We use the formula $u_n = a + (n-1)d$

$$16 = a + 4d \text{ (1)}$$
$$30 = a + 9d \text{ (2)}$$

Let's solve these equations simultaneously. Subtracting (1) from (2),

$$14 = 5d$$
$$d = \frac{14}{5}$$

Substitute $d = \frac{14}{5}$ back into (1).

$$16 = a + 4\left(\frac{14}{5}\right)$$
$$a = \frac{24}{5}$$

ii.

We want $s_n > 1000$. Let's use the formula

$$\frac{n}{2}(2a + (n-1)d) > 1000$$
$$\frac{n}{2}\left(2\left(\frac{24}{5}\right) + (n-1)\left(\frac{14}{5}\right)\right) > 1000$$
$$\frac{n}{2}\left(\frac{48}{5} + \frac{14}{5}n - \frac{14}{5}\right) > 1000$$
$$\frac{n}{2}\left(\frac{34}{5} + \frac{14}{5}n\right) > 1000$$
$$\frac{34}{10}n + \frac{14}{10}n^2 > 1000$$
$$7n^2 + 17n - 5000 > 0$$

Using the quadratic formula,

$$x < \frac{-17 - \sqrt{140289}}{14} \approx -27.97 \text{ } or \text{ } x > \frac{-17 + \sqrt{140289}}{14} \approx 25.54$$

Since x cannot be negative, $x > 25.54$. You need at least 26 terms for the sum to exceed 1000.

37)

This is an arithmetic sequence. We have $a = 4, d = 9 - 4 = 5$.

We want $s_n > 2000$. Let's use the formula.

$$\frac{n}{2}(2(4) + (n-1)(5)) > 2000$$
$$\frac{n}{2}(8 + 5n - 5) > 2000$$
$$\frac{n}{2}(3 + 5n) > 2000$$
$$\frac{3}{2}n + \frac{5}{2}n^2 - 2000 > 0$$
$$5n^2 + 3n - 4000 > 0$$

Using the quadratic formula,

$$n < \frac{-3 - \sqrt{80009}}{10} \text{ or } n > \frac{-3 + \sqrt{80009}}{10}$$

Since n cannot be negative, $n > \frac{-3+\sqrt{80009}}{10} \approx 27.99$. The least n satisfying this inequality is 28.

2.2.4 Worded Questions

38)

i.

If she earns 500 more every year, it'll take her $(25000 - 20000)/500 = 10$ years to reach her maximum salary. So, for the first ten years the amount she earns every year is an arithmetic sequence with initial earning $a = 20000$, and the common difference $d = 500$.

So the total she ears in the first n years must be $s_n = \frac{n}{2}(2a + (n-1)d), n \leq 10$.

$$s_{10} = \frac{10}{2}(2(20000) + (10-1)500)$$
$$s_{10} = 222500$$

The total she earns is £222500.

ii.

Over the next five years Carols has no salary rise, so she earns 25000 every year for five more years.

$$222500 + 5 \times 25000 = 347500$$

The total she earns over 15 years is £347500

iii.

It is unlikely that her salary will increase by the same amount each year.

39)

i.

This amount of mass the company extracts per year starting from 2018 is a geometric sequence, with starting value $a = 4500$ and common ratio 0.98.

The total mass extracted between 2018 and 2040, inclusive, is the sum of the first $2040 - 2018 + 1 = 23$ terms, ie s_{23}.

$$s_{23} = \frac{4500(1 - 0.98^{23})}{1 - 0.98} = 83621.86152 \ldots \approx 83600 \text{ tonnes}$$

ii.

The mass of the mineral extracted in 2040 is simply the $23rd$ term of the geometric sequence, by our previous calculation.

$$u_{23} = 4500(0.98)^{23-1} = 2885.268312 \approx 2890 \text{ tonnes}$$

iii.

We can see that in every year between 2018 and 2040, the company extracts more than 1500 tonnes of mineral, since even in 2040 the company is still extracting $2890 > 1500$ tonnes.

So, we know that the company paid £800 per tonne for 1500 tonnes every year for 23 years, and the rest are paid in £600 per tonne.

$$800 \times (1500 \times 23) + 600 \times (83621.86152 - 1500 \times 23) = 57073116.91$$
$$x \approx 57100000$$

40)

i.

In Model A, we have an arithmetic sequence that models the amount of batteries made per year.
We have $a = u_1 = 2600$, $u_{10} = 12000$. We can use $u_n = a + (n-1)d$ to work out the common difference.

$$u_{10} = 12000 = 2600 + (10-1)d$$
$$9400 = 9d$$
$$d = 9400/9$$

So, the number of batteries the company makes in year 2 would be

$u_2 = 2600 + (2-1)\left(\frac{9400}{9}\right) = 3644.44444 \approx 3644$ under model A.

Note: accept 3645 also

ii.

In Model B, we have an geometric sequence that models the amount of batteries made per year. We have $a = v_1 = 2600$, $v_{10} = 12000$. We can use $v_n = ar^{n-1}$ to work out the common ratio.

$$v_{10} = 12000 = 2600r^{10-1}$$
$$\frac{60}{13} = r^9$$
$$r = \sqrt[9]{\frac{60}{13}}$$

So, the number of batteries the company makes in year 2 would be $v_2 =$ $2600\left(\sqrt[9]{\frac{60}{13}}\right)^{2-1} = 3081.585526 \ldots \approx 3080$ batteries under model B.

iii.

We calculate $s_{10} = u_1 + u_2 + \cdots + u_{10}$ using the formula $s_n = \frac{n}{2}(a + l)$.

$$s_{10} = \frac{10}{2}(2600 + 12000) = 73000$$

We calculate $t_{10} = v_1 + v_2 + \cdots + v_{10}$ using the formula $t_n = \frac{a(1-r^n)}{1-r}$

$$t_{10} = \frac{2600\left(1 - \left(\sqrt[9]{\frac{60}{13}}\right)^{10}\right)}{1 - \sqrt[9]{\frac{60}{13}}} = 62749.03356$$

The difference is

$$73000 - 62749.03356 = 10250.96644 \approx 10300$$

Note: Accept 10,200 batteries also

41)

i.

This arithmetic sequence has $a = u_1 = 140$, and $d = 160 - 140 = 20$.
Plugging into the formula

$$u_{20} = 140 + (20-1)20 = 520$$

ii.

Using the formula $s_n = \frac{n}{2}(a + l)$

$$s_{20} = \frac{20}{2}(140 + 520) = 6600$$

iii.

We can use the formula

$$s_n = \frac{n}{2}(a + l)$$
$$8500 = \frac{n}{2}(300 + 700)$$

$$8500 = \frac{n}{2}1000$$
$$8500 = 500n$$
$$n = 17$$

2.2.4.1 Being Careful with difference between years

42)

i.

The salary each year is an arithmetic sequence.
We have $a = 17000, d = 1500$. We want to solve for k, where $u_k = 32000$

$$32000 = u_k = a + (k-1)d = 17000 + (k-1)1500$$
$$15000 = (k-1)1500$$
$$10 = k - 1$$
$$k = 11$$

ii.

For the first 11 years, Jess has earned s_{11} pounds.

$$s_{11} = \frac{11}{2}(17000 + 32000) = \ 269500$$

For the last 9 years, Jess ears £32000 each year.

The total is $269500 + 32000 \times 9 = £557500$.

43)

i.

John gets a gift worth £60 on his tenth birthday, £75 on his eleventh, £90 on his twelfth.
So, the total he gets is

$$£60 + £75 + £90 = £225$$

ii.

The value of the gift each year is an arithmetic sequence.
We have $a = 60, d = 15$.
The eighteenth birthday is the $18 - 10 + 1 = 9th$ year he receives gifts from his uncle.

$$u_9 = a + (9-1)d$$
$$u_9 = 60 + 8(15) = £180.$$

iii.

The twenty-first birthday is the $21 - 10 + 1 = 12th$ year he receives gifts from his uncle.
The total value of gifts is s_{12}

$$s_{12} = \frac{12}{2}\big(2(60) + (12-1)(15)\big) = \ £1710$$

iv.

We have $s_n = 3375$

$$3375 = s_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}\big(2(60) + (n-1)(15)\big)$$
$$3375 = \frac{n}{2}(120 + 15n - 15)$$
$$3375 = \frac{n}{2}(105 + 15n)$$
$$3375 = \frac{105}{2}n + \frac{15}{2}n^2$$
$$225 = \frac{7}{2}n + \frac{1}{2}n^2$$
$$n^2 + 7n = 450 = 25 \times 18$$

v.

We have $n^2 + 7n = 25 \times 18$

$$n(n+7) = 25 \times 18$$

We deduce that

$$n = 18 \ or - 25$$

Clearly $n = 18$ is the only answer that makes sense here. This means he was 27 years old.

44)

i.

The number of mobile phones produced each week forms an arithmetic sequence. We have $a = 200, d = 20$.
We solve for N, where $u_N = 600$.

$$600 = u_N = 200 + (N-1)20$$
$$400 = (N-1)20$$
$$N - 1 = 20$$
$$N = 21$$

ii.

Hint: Find s_{21} and then from week 22 to 52 i.e. 31 weeks multiply by 600

For the first 21 weeks, the company makes s_{21} phones in total.

$$s_{21} = \frac{21}{2}(200 + 600) = 8400$$

Then for the next $52 - 21 = 31$ weeks, the company makes 600 phones each week. The total produced is therefore

$$8400 + 31 \times 600 = 27000$$

45)

i.

Hint: This is the $13th$ year $(2002 - 1989 \text{ or } 2002 - 1989 + 1)$

The number of bicycles produced each week forms an arithmetic sequence. We have $a = 140$, $u_{12} = 206$.

We can solve for d using the formula for u_n.

$$206 = u_{12} = 140 + (12-1)d$$
$$66 = 11d$$
$$d = 6$$

i.

During the first 12 weeks, the company produced s_{12} bikes.

$$s_{12} = \frac{12}{2}(140 + 206) = 2076$$

For the remaining $52 - 12 = 10$ weeks, the company produced 206 bikes per week every week. The total is therefore

$$2076 + 206 \times 10 = 4136$$

46)

i.

We have $a = 160, r = \frac{240}{160} = \frac{3}{2}$. Since 2002 is the $2002 - 1990 + 1 = 13th$ year, we calculate u_{13}

$$u_{13} = ar^{13-1} = 160\left(\frac{3}{2}\right)^{12} = 20759.41406 \approx 20759$$

ii.

We calculate $u_n > 5000$.

$$5000 < u_n = 160\left(\frac{3}{2}\right)^{n-1}$$
$$5000 < 160\left(\frac{3}{2}\right)^{n-1}$$
$$\frac{125}{4} < \left(\frac{3}{2}\right)^{n-1}$$

We take log of both sides, then divide by $\ln\left(\frac{3}{2}\right) > 0$, so we don't need to reverse the inequality.

$$\ln\frac{125}{4} < (n-1)\ln\left(\frac{3}{2}\right)$$
$$\frac{\ln\left(\frac{125}{4}\right)}{\ln\left(\frac{3}{2}\right)} < n - 1$$
$$n < 9.489\ldots$$

So, we have $n = 10$ is the year units sold first exceeds 5000.

Which is year $1990 + 10 - 1 = 1999$.

iii.

We already calculated 2002 is the $13th$ year. So, this value is just s_{13}. We use the formula for geometric series.

$$s_{13} = \frac{160\left(1-\left(\frac{3}{2}\right)^{13}\right)}{1-\left(\frac{3}{2}\right)} = 61958.24219 \approx 61958$$

3 Gold

3.1 Without Formulae

47)

i.

We use the fact that they have a common ratio.

$$\frac{\frac{2k+3}{4}}{\frac{k-3}{2}} = \frac{\frac{12k+3}{8}}{\frac{2k+3}{4}}$$

$$\frac{2k+3}{4} \times \frac{2k+3}{4} = \frac{12k+3}{8} \times \frac{k-3}{2}$$

$$16(2k+3)^2 = 16(12k+3)(k-3)$$

$$(2k+3)^2 = (12k+3)(k-3)$$

$$4k^2 + 12k + 9 = 12k^2 - 33k - 9$$

$$8k^2 - 45k - 18 = 0$$

ii.

We solve for k.

$$(8k+3)(k-6) = 0$$

$$k = -\frac{3}{8} \; or \; 6$$

We split into two cases.

Case $k = -\frac{3}{8}$	**Case $k = 6$**
The sequence becomes $-\frac{27}{16}, \frac{9}{16}, -\frac{3}{16}$ We have common ratio $r = \frac{\frac{9}{16}}{-\frac{27}{16}} = -\frac{1}{3}$ So, the first 5 terms are $-\frac{27}{16}, \frac{9}{16}, -\frac{3}{16}, \frac{1}{16}, -\frac{1}{48}$	The sequence becomes $\frac{3}{2}, \frac{15}{4}, \frac{75}{8}$ We have common ratio $r = \frac{\frac{15}{4}}{\frac{3}{2}} = \frac{5}{2}$ So, the first 5 terms are $\frac{3}{2}, \frac{15}{4}, \frac{75}{8}, \frac{375}{16}, \frac{1875}{32}$

48)

If $p, q, 10$ are in arithmetic progression, then they must have a common difference.

$$q - p = 10 - q$$

$$q = \frac{10+p}{2} \quad ①$$

If $q, p, 10$ are in geometric progression, then they must have a common ratio.

$$\frac{p}{q} = \frac{10}{p}$$

$$p^2 = 10q \ ②$$

We can solve ① and ② simultaneously. We substitute ① into ②.

$$p^2 = \frac{10(10+p)}{2}$$

$$p^2 = 50 + 5p$$

$$p^2 - 5p - 50 = 0$$

$$(p-10)(p+5) = 0$$

$$p = 10 \ or - 5$$

If we substitute these two values of p back into ①, we get $p = 10, q = 10$ or $p = -5, q = \frac{5}{2}$. Since p, q are distinct numbers, $p = -5, q = \frac{5}{2}$.

3.2 Using Formulae

3.2.1 Simultaneous Equations

49)

We have $u_2 + u_3 = -12, u_3 + u_4 = -36$. Using the formula $u_n = ar^{n-1}$, we get

$$ar + ar^2 = -12 \ ①$$

$$ar^2 + ar^3 = -36 \ ②$$

Dividing ① by ② gives (we can do this because neither is 0). Note you can also factorise each and then use substitution instead.

$$\frac{ar + ar^2}{ar^2 + ar^3} = \frac{-12}{-36}$$

$$\frac{r + r^2}{r^2 + r^3} = \frac{1}{3}$$

$$\frac{r(1+r)}{r^2(1+r)} = \frac{1}{3}$$

$$\frac{1}{r} = \frac{1}{3}$$

$$r = 3$$

50)

We have $u_1 + u_2 = 8$, $u_3 + u_4 = 2$. Using the formula $u_n = ar^{n-1}$we get

$$a + ar = 8 \ ①$$

$$ar^2 + ar^3 = 2 \ ②$$

Dividing ① by ② gives (we can do this because neither is 0)

$$\frac{a + ar}{ar^2 + ar^3} = \frac{8}{2}$$

$$\frac{1 + r}{r^2 + r^3} = 4$$

$$\frac{(1+r)}{r^2(1+r)} = 4$$

$$\frac{1}{r^2} = 4$$

$$r = \pm\frac{1}{2}$$

Substituting $r = \pm\frac{1}{2}$ back into ① gives

$$a + \frac{a}{2} = 8$$
$$\frac{3}{2}a = 8$$
$$a = \frac{16}{3}$$
$$a - \frac{a}{2} = 8$$
$$\frac{1}{2}a = 8$$
$$a = 16$$

So, $r = \frac{1}{2}, a = \frac{16}{3}$ or $r = -\frac{1}{2}, a = 16$.

51)

We have $u_1 + u_2 = 50,\ u_2 + u_3 = 30$. Using the formula $u_n = ar^{n-1}$ we get

$$a + ar = 50 \quad ①$$
$$ar + ar^2 = 30 \quad ②$$

Dividing ① by ② gives (we can do this because neither is 0)

$$\frac{a + ar}{ar + ar^2} = \frac{50}{30}$$
$$\frac{1 + r}{r + r^2} = \frac{5}{3}$$
$$\frac{(1 + r)}{r(1 + r)} = \frac{5}{3}$$
$$\frac{1}{r} = \frac{5}{3}$$
$$r = \frac{3}{5}$$

Substituting $r = \frac{3}{5}$ back into ① gives

$$a + \frac{3}{5}a = 50$$
$$\frac{8}{5}a = 50$$
$$a = \frac{250}{8} = \frac{125}{4}$$

Since $-1 < r = \frac{3}{5} < 1$, we can substitute a, r into the formula for sum to infinity to get

$$s_\infty = \frac{a}{1 - r} = \frac{\frac{125}{4}}{1 - \frac{3}{5}} = \frac{625}{8}$$

52)

We know their sum to infinities equal, so let's equate the two sums to infinities.

$$\frac{3a}{1 - r} = \frac{a}{1 - (-2r)}$$
$$(1 + 2r)3a = a(1 - r)$$

Divide both sides by a. This assumes that $a \neq 0$, if this is the case then both geometric sequences would be just $0, 0, 0, \ldots$ then any common ratio suffices.

$$3(1 + 2r) = 1 - r$$
$$3 + 6r = 1 - r$$
$$7r = -2$$
$$r = -\frac{2}{7}$$

53)

We know the formula $s_n = \frac{a(1-r^n)}{1-r}$, let's substitute this into $s_{10} = 4s_5$.

$$\frac{a(1-r^{10})}{1-r}=\frac{4a(1-r^5)}{1-r}$$

Cancelling terms out, we get

$$1-r^{10}=4(1-r^5)$$
$$1-r^{10}=4-4r^5$$
$$r^{10}-4r^5+3=0$$

This is a hidden quadratic, set $x=r^5$.

$$x^2-4x+3=0$$
$$(x-3)(x-1)=0$$
$$x=3\ or\ 1$$

Since $r\neq 1$ (because our formula for geometric series would end up dividing by 0), $r=\sqrt[5]{3}$.

54)

We have the fourth term, ar^3, is eight times the first term, a.

$$ar^3=8a$$
$$r^3=8$$
$$r=2$$

We also have $s_{10}=2557.5$

$$2557.5=\frac{a(1-r^{10})}{1-r}=\frac{a(1-2^{10})}{1-2}$$
$$2557.5=1023a$$
$$a=\frac{5}{2}$$

55)

We have $u_1=u_2+9, s_\infty=81$

Let's use the formula for u_n and s_∞

$s_\infty=81$ The formula tells us that $\frac{a}{1-r}=81$	$u_1=u_2+9$ The formula tells us that $a=ar+9$

We have

$$\frac{a}{1-r}=81 \quad (1)$$
$$a=ar+9 \quad (2)$$

Rearranging (1), we get $a=81(1-r)$. Sub it into (2).

$$81(1-r)=81(1-r)r+9$$
$$81-81r=81r-81r^2+9$$
$$81r^2-162r+72=0$$
$$9r^2-18+8=0$$
$$(3r-4)(3r-2)=0$$
$$r=\frac{4}{3} or \frac{2}{3}$$

But this is an infinite geometric series, so $-1<r<1$. Therefore, $r=\frac{2}{3}$.

56)

We have $u_8=3u_3$. Let's use the formula for u_n.

$$a+7d=3(a+2d)$$
$$a+7d=3a+6d$$
$$2a=d\ (*)$$

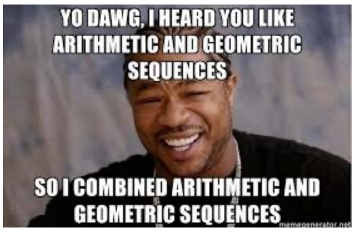
Let's calculate the sum of first eight terms, s_8, and the sum of first four terms s_4.

s_8	s_4

$s_8 = \frac{8}{2}(2a + 7d)$ $s_8 = 4(2a + 7d)$ $s_8 = 8a + 28d$ Let's substitute the equation $(*)$ in. $s_8 = 8a + 28(2a)$ $s_8 = 64a$	$s_4 = \frac{4}{2}(2a + 3d)$ $s_4 = 2(2a + 3d)$ $s_4 = 4a + 6d$ Let's substitute the equation $(*)$ in. $s_4 = 4a + 6(2a)$ $s_4 = 16a$

We have $64a = 4(16a)$, ie $s_8 = 4s_4$.

3.2.2 Arithmetic and Geometric Together In One Question



57)

i.

Let the geometric sequence by u_n, the arithmetic sequence be v_n.
We have

$$u_1 = v_1 = 8 = a$$

Both series have the same first term, so they can have a common a. Then, we have

$$u_2 = v_9$$
$$u_3 = v_{21}$$

We substitute in the formula for geometric sequence and arithmetic sequence on each side.

$$8r = 8 + 8d \text{ ①}$$
$$8r^2 = 8 + 20d \text{ ②}$$

Let's solve these equations simultaneously.
We simplify and rearrange ① to get

$$r = 1 + d$$
$$d = r - 1 \text{③}$$

We substitute this into ②.

$$8r^2 = 8 + 20(r - 1)$$
$$8r^2 = 8 + 20r - 20$$
$$8r^2 - 20 + 12 = 0$$
$$2r^2 - 5 + 3 = 0$$
$$(2r - 3)(r - 1) = 0$$
$$r = \frac{3}{2} or\ 1$$

We're told that $r \neq 1$, so $r = \frac{3}{2}$.

ii.

The geometric sequence: $u_4 = ar^3 = 8\left(\frac{3}{2}\right)^3 = 27$

For the arithmetic sequence, we need to work out the common difference d. We use equation ③ $d = r - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

The arithmetic sequence: $v_4 = a + 3d = 8 + 3\left(\frac{1}{2}\right) = \frac{19}{2}$.

58)

i.

Let the geometric sequence by u_n, the arithmetic sequence be v_n.
We have

$$u_1 = v_1 = 2 = a$$

Both series have the same first term, so they can have a common a. Then, we have

$$u_2 = v_2$$
$$u_3 = v_5$$

We substitute in the formula for geometric sequence and arithmetic sequence on each side.

$$2r = 2 + d \text{ ①}$$
$$2r^2 = 2 + 4d \text{ ②}$$

Let's solve these equations simultaneously.
We rearrange ① to get

$$r = 1 + \frac{d}{2} \text{ ③}$$

We substitute this into ②.

$$2r^2 = 2 + 4d$$
$$2\left(1 + \frac{d}{2}\right)^2 = 2 + 4d$$
$$2 + 2d + \frac{d^2}{2} = 2 + 4d$$
$$\frac{d^2}{2} - 2d = 0$$
$$d^2 - 4d = 0$$
$$d(d - 4) = 0$$
$$d = 0 \text{ or } 4$$

We may assume that $d \neq 0$. So, $d = 4$

ii.

We substitute d back into equation ③.

$$r = 1 + \frac{4}{2} = 3$$

59)

i.

We have $a = u_1 = 27$.
We also have the sum to infinity $s_\infty = \frac{81}{2} = \frac{a}{1-r}$

Substitute $a = 27$.

$$\frac{81}{2} = \frac{27}{1-r}$$
$$81(1 - r) = 54$$
$$1 - r = \frac{54}{81}$$
$$r = 1 - \frac{54}{81} = \frac{27}{81} = \frac{1}{3}$$

ii.

We have the formula $v_n = b + (n-1)d$, where b is the first term, d is the common difference. We use b instead of a here because a is already used for the first term of the geometric sequence. We do not know if they are the same.

We have $v_2 = u_2$, $v_4 = u_4$

$$b + d = ar = 9$$
$$b + 3d = ar^3 = 1$$

Solving the equations simultaneously, we get $d = -4, b = 13$.

Now, $\sum_{n=1}^{N} v_n = s_n = \frac{n}{2}(2b + (n-1)d)$

$$s_n = \frac{n}{2}\big(2(13) + (n-1)(-4)\big)$$
$$= \frac{n}{2}(26 - 4n + 4)$$
$$= \frac{n}{2}(30 - 4n)$$
$$= 15n - 2n^2$$

We want $s_n > 0$.

$$15n - 2n > 0$$

$$2n^2 - 15n < 0$$
$$n(2n - 15) < 0$$
$$0 < n < \frac{15}{2}$$

Since n has to be an integer, the greatest n is 7.

60)

i. We have, from the formula, $u_n = 1.6 + 1.5(n-1) = 1.5n + 0.1$, $v_N = 3r^{n-1}$.
So,
$$u_n - v_n = 1.5n + 0.1 - 3(1.2)^{n-1}$$

ii. If $u_n > v_n$, $u_n - v_n > 0$.
$$1.5n + 0.1 - 3(1.2)^{n-1} > 0$$
This can only be solved via trial or error or graphically
$3 \le n \le 9$

iii. This should be solved graphically
1.64

61)

i. Substitute the formulas for an arithmetic sequence, we have
$$a + 6d, a + 2d, a$$
Form a geometric sequence.

By the common ratio property, we have
$$\frac{a+2d}{a+6d} = \frac{a}{a+2d}$$
$$(a+2d)^2 = a(a+6d)$$
$$a^2 + 4ad + 4d^2 = a^2 + 6ad$$
$$4d^2 - 2ad = 0$$
$$2d(2d - a) = 0$$
Since the common difference d is non-zero, we have $d = \frac{a}{2}$

ii. We have the seventh term $a + 6d = 3$
$$a + 6\left(\frac{a}{2}\right) = 3$$
$$4a = 3$$
$$a = \frac{3}{4}$$
Substitute this back into $d = \frac{a}{2}$ gives us $d = \frac{3}{8}$.

We also have $r = \frac{a}{a+2d} = \frac{1}{2}$.

Now we have the sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms of the geometric sequence by 200.
$$\frac{n}{2}\left(2\left(\frac{3}{4}\right) + (n-1)\left(\frac{3}{8}\right)\right) - \frac{\left(\frac{3}{4}\right)\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \left(\frac{1}{2}\right)} \ge 200$$
$$\frac{9}{16}n + \frac{3}{16}n^2 - \frac{3}{8} + \frac{3}{8}\left(\frac{1}{2}\right)^n \ge 200$$
We can solve this using a calculator or via trial and error. We have $n \ge 31.68$. The smallest integer n is 32.

62)

Given formula for s_n, we can work out $a = u_1 = s_1$.
$$a = 4(1)^2 - 2(1) = 2$$
We can work out $u_2 = s_2 - s_1 = s_2 - a$
$$u_2 = 4(2)^2 - 2(2) - 2 = 10$$
So, $d = u_2 - u_1 = 10 - 2 = 8$.

Now we have a general formula for $u_n = a + (n-1)d = 2 + (n-1)8 = 8n - 6$.

Now, we can use the common ratio formula to find r.

$$\frac{u_m}{u_2} = r$$
$$\frac{u_{32}}{u_m} = r$$

Multiply the above together, we get

$$\frac{u_m}{u_2} \times \frac{u_{32}}{u_m} = \frac{u_{32}}{u_2} = r^2$$

We use the formula for u_n

$$\frac{8(32) - 6}{8(2) - 6} = r^2$$
$$25 = r^2$$
$$r = \pm 5$$

We know

$$\frac{u_m}{u_2} = r = \pm 5$$

So,

$$u_m = \pm 5(8(2) - 6) = \pm 50$$

We find m by plugging it back into the formula for u_n.

$$\pm 50 = 8m - 6$$

$50 = 8m - 6$ $56 = 8m$ $m = 7$	$-50 = 8m - 6$ $-44 = 8m$ $m = -\frac{11}{2}$

Since m must be a positive integer, $m = 7$.

3.2.3 Worded

63)

i. We have an arithmetic sequence with the first term A, and common difference $d + 1$. So, the formula is $u_n = A + (n - 1)(d + 1)$

Plug in $n = 14$.

$$u_{14} = A + (14 - 1)(d + 1)$$
$$u_{14} = A + 13d + 13$$

ii. Let Yi's running time be v_n. Then this arithmetic sequence has firs term $A - 13$, common difference $2d - 1$. The formula is

$$v_n = A - 13 + (n - 1)(2d - 1)$$

When $n = 14$,

$$v_{14} = A - 13 + 13(2d - 1) = A + 26d - 26$$

We are given that $u_{14} = v_{14}$

$$A + 13d + 13 = A + 26d - 26$$
$$13d + 13 = 26d - 26$$
$$39 = 13d$$
$$d = 3$$

iii. We have $s_{14} = 784$. Substituting the first term and the common difference into the equation for arithmetic series, we get

$$s_n = \frac{n}{2}\left(2A + (n - 1)(d + 1)\right)$$
$$784 = s_{14} = \frac{14}{2}\left(2A + (14 - 1)(3 + 1)\right)$$
$$784 = 7(2A + 52)$$
$$112 = 2A + 52$$
$$A = 30$$

64)

i.

We have an arithmetic sequence with initial term $a = 1500$, common difference $d = -x$. The total sales during the first six months is £8100. So $s_6 = 8100$. We substitute this value into the arithmetic series formula to find out x.

$$s_n = \frac{n}{2}(2a + (n-1)d)$$
$$s_6 = \frac{6}{2}(2(1500) + (6-1)(-x))$$
$$8100 = 3(3000 - 5x)$$
$$2700 = 3000 - 5x$$
$$-300 = -5x$$
$$x = 60$$

ii.

We substitute $a = 1500, d = -60$ into the formula for n^{th} term.

$$u_n = a + (n-1)d$$
$$u_8 = 1500 + (8-1)(-60)$$
$$= 1500 - 420$$
$$= 1080$$

The sales in the eighth month is £1080.

iii.

The expected value of sales in pounds during the first n months is just the sum s_n.

$$s_n = \frac{n}{2}(2a + (n-1)d)$$
$$s_n = \frac{n}{2}(2(1500) + (n-1)(-60))$$
$$s_n = \frac{n}{2}(3000 - 60n + 60)$$
$$s_n = \frac{n}{2}(3060 - 60n)$$
$$s_n = \frac{n}{2}(60)(51 - n)$$
$$s_n = 30n(51 - n)$$
$$k = 30$$

iv.

This model cannot be valid for a long period of time because eventually the sales in a month would become negative, which is not possible.

65)

i.

The pay each day forms an arithmetic sequence with initial term a, common difference d.

$$u_n = a + (n-1)d$$

We know a picker earns £40.75 on their $30th$ day. $u_{30} = 40.75$

$$40.75 = a + 29d \quad (1)$$

ii.

We are given that a picker who works for all 30 days will earn £1005 in total, hence $s_{30} = 1005$

Since we already know the value for the $30th$ term. We can use the formula

$$s_n = \frac{n}{2}(a + l)$$
$$s_{30} = \frac{30}{2}(a + 40.75)$$
$$1005 = 15(a + 40.75) \quad (2)$$

iii.

We can solve a from equation (2) in the previous question.

$$67 = a + 40.75$$
$$a = 26.25$$

Put this back into (1)

$$40.75 = 26.25 + 29d$$
$$14.5 = 29d$$

$$d = \frac{1}{2}$$
$$50\,p$$

66)

i.

The amount Shelim earns each year forms an arithmetic sequence with $a = 14000$, $d = 1500$ (until he reaches £26000).

We can calculate how much she will earn in year 9 by calculating u_9, and making sure this number is not more than 26000.

$$u_9 = a + 8d$$
$$u_9 = 14000 + 8(1500) = 26000$$

So Shelim will earn £26000 in year 9, and his salary will not grow after that.

ii.

We use the formula $s_n = \frac{n}{2}(a + l)$, we can do this because the first 9 years his salary per year forces an arithmetic sequence.

$$s_9 = \frac{9}{2}(14000 + 26000) = 180000$$

The answer is £180000

iii.

Anna's salary for the first 10 years forms an arithmetic sequence v_n with starting salary A, and common difference 1000. We can work out A by using our knowledge that she earns £26000 in year 10.

$$v_n = a + (n-1)d$$
$$26000 = A + (10-1)1000$$
$$26000 = A + 9000$$
$$A = 17000$$

We label the corresponding sums of this sequence v_n to be t_n. Now, we can plug these values into the formula $t_n = \frac{n}{2}(a + l)$ to work out how much Anna will earn for the first 10 years.

$$t_{10} = \frac{10}{2}(17000 + 26000) = 215000$$

So, Anna will earn £215000 in the first 10 years.

Shelim earns £180000 in the first 9 years, he will earn another £26000 in year 10, bringing the total to £180000 + £26000 = £206000.

So, the total difference is

$$£206000 - £215000 = £9000.$$

3.2.4 With Logs

67)

We have the first term $\ln a$, and the common difference is $\ln 3$. We can use our knowledge of the $13th$ term to work out the value of a.

$$8\,ln\,9 = u_{13} = \ln a + (13-1)\ln 3$$
$$8\ln 9 = \ln a + 12\ln 3$$
$$8\,ln\,9 - 12\,ln\,3 = \ln a$$
$$ln\,9^8 - ln\,3^{12} = \ln a$$

$$\ln\left(\frac{9^8}{3^{12}}\right) = \ln a$$
$$\frac{9^8}{3^{12}} = a$$
$$a = \frac{9^8}{3^{12}} = \frac{9^8}{9^6} = 81$$

68)

i.

$$r = \frac{\log_2 x}{2\log_2 x} = \frac{1}{2}$$

ii.

We have $a = 2\log_2 x, r = \frac{1}{2}$

$$S_\infty = \frac{a}{1-r} = \frac{2\log_2 x}{1-\frac{1}{2}} = 4\log_2 x$$

iii.

$$d = \log_2\frac{x}{2} - \log_2 x = \log_2\left(\frac{\frac{x}{2}}{x}\right) = \log_2\frac{1}{2} = -1$$

iv.

We have $a = \log_2 x, d = -1$, we use the formula for S_{12}.

$$S_{12} = \frac{12}{2}(2\log_2 x + (12-1)(-1))$$
$$S_{12} = 6(2\log_2 x - 11)$$
$$S_{12} = 12\log_2 x - 66$$

v.

We have that $S_{12} = \frac{S_\infty}{2}$. Substitute our values in.

$$12\,log_2\, x - 66 = \frac{4\,log_2\, x}{2}$$
$$12\log_2 x - 66 = 2\log_2 x$$
$$10\log_2 x = 66$$
$$\log^2 x = 6.6$$
$$x = 2^{6.6}$$

3.2.5 With Inequalities and Logs

69)

This series has $a = 3, r = \frac{6}{3} = 2$.

We want $s_n > 100000$. Let's use the formula.

$$\frac{a(r^n - 1)}{r-1} > 100000$$
$$\frac{3(2^n - 1)}{2-1} > 1000000$$
$$3(2^n) - 3 > 1000000$$
$$3(2^n) > 99997$$
$$2^n > \frac{99997}{3}$$
$$n > \log_2\frac{99997}{3} \approx 15.02$$

We need at least 16 terms.

70)

$u_2 = 192$

$$ar = 192$$

$u_3 = 144$

$$ar^2 = 144$$

Solving these simultaneously

$$a = \frac{192}{r}$$

$$\frac{192}{r}r^2 = 144$$

$$r = \frac{144}{192} = \frac{3}{4}$$

$$a\left(\frac{3}{4}\right) = 192$$

$$a = \frac{192}{\frac{3}{4}} = 256$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

We need to solve when the sum exceeds 1000

$$\frac{256\left(1-\frac{3}{4}^n\right)}{1-\frac{3}{4}} > 1000$$

$$\frac{256\left(1-\frac{3}{4}^n\right)}{\frac{1}{4}} > 1000$$

$$256\left(1-\frac{3}{4}^n\right) > 250$$

$$1-\frac{3}{4}^n > \frac{250}{256}$$

$$-\frac{3}{4}^n > \frac{250}{256} - 1$$

$$-\frac{3}{4}^n > -\frac{3}{128}$$

Note: we swapped the inequality sign since we divide by a negative

$$\frac{3}{4}^n < \frac{3}{128}$$

$$log\frac{3}{4}^n < log\frac{3}{128}$$

$$n\log\frac{3}{4} < log\frac{3}{128}$$

$$n > \frac{log\frac{3}{128}}{log\frac{3}{4}}$$

Note: we swapped the inequality sign since $log\frac{3}{4}$ is less than zero

$$n > 13.05$$

$$n = 14$$

71)

$$a = 120$$

$$S_\infty = \frac{120}{1-r} = 480$$

$$120 = 480(1-r)$$

$$120 = 480 - 480r$$

$$480r = 360$$

$$r = \frac{3}{4}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

We need to solve when the sum is greater than 300

$$\frac{120(1-\frac{3}{4}^n)}{1-\frac{3}{4}} > 300$$

$$\frac{120\left(1-\frac{3}{4}^n\right)}{\frac{1}{4}} > 300$$

$$120\left(1-\frac{3}{4}^n\right) > 75$$

$$1-\frac{3}{4}^n > \frac{75}{120}$$

$$-\frac{3}{4}^n > \frac{75}{120} - 1$$

$$-\frac{3}{4}^n > -\frac{3}{8}$$

Note: we swapped the inequality sign since we divide by a negative

$$\frac{3}{4}^n < \frac{3}{8}$$

$$log\frac{3}{4}^n < \log\frac{3}{8}$$

$$n\log\frac{3}{4} < \log\frac{3}{8}$$

$$n > \frac{log\frac{3}{8}}{log\frac{3}{4}}$$

Note: we swapped the inequality sign since $log\frac{3}{4}$ is less than zero

$$n > 3.409$$

$$n = 4$$

72)

i.

We have $a = u_1 = \frac{1}{81}$. We can use our formula and knowledge of u_4 to work out r.

$$\frac{1}{3} = u_4 = \frac{1}{81}r^3$$
$$27 = r^3$$
$$r = 3$$

ii.

We can use the formula for s_n.

$$\frac{a(r^n - 1)}{r - 1} > 40$$
$$\frac{\frac{1}{81}(3^n - 1)}{3 - 1} > 40$$
$$\frac{3^n - 1}{162} > 40$$
$$3^n - 1 > 6480$$
$$3^n > 6481$$
$$n > \log_3 6481 \approx 7.99$$

So, the smallest value of n satisfying the above would be $n = 8$.

73)

i.

We have $r = \frac{u_2}{u_1} = \frac{1.6}{0.64} = \frac{5}{2}$

ii.

We know that $a = u_1 = 0.64$, substitute a, r into the formula for s_n

$$s_6 = \frac{0.64\left(\left(\frac{5}{2}\right)^6 - 1\right)}{\frac{5}{2} - 1} = 103.74$$

iii.

Let's substitute the formula for s_n into the inequality and solve for n.

$$s_n > 7500$$
$$\frac{0.64\left(\left(\frac{5}{2}\right)^n - 1\right)}{\frac{5}{2} - 1} > 75000$$
$$\frac{32}{75}\left(\left(\frac{5}{2}\right)^n - 1\right) > 75000$$
$$\left(\frac{5}{2}\right)^n - 1 > 175781.25$$
$$\left(\frac{5}{2}\right)^n > 175182.25$$
$$n \ln\left(\frac{5}{2}\right) > \ln 175182.25$$

Divide both sides by $\ln\frac{5}{2}$, we don't need to reverse the inequality because $\frac{5}{2} > 1$.

$$n > \frac{\ln 175181.25}{\ln\frac{5}{2}} \approx 13.18$$

The least value of n is therefore 14.

74)

The amount of money Carlos saves each year forms a geometric sequence u_n with $a = 100, r = 1.1$.

The total money he has saved would be s_n after n years, so we want $s_n > 1000$

$$\frac{100(1 - 1.1^n)}{1 - 1.1} > 1000$$
$$-1000(1 - 1.1^n) > 1000$$
$$-1000 + 1000(1.1^n) > 1000$$
$$1000(1.1^n) > 2000$$
$$1.1^n > 2$$
$$n > \log_{1.1} 2 \approx 7.27$$

So after 8 years Carlos' saving would exceed £1000.

75)

i.

We have $a = 5, r = \frac{4}{5}, s_k > 24.95$. Let's substitute the formula for s_k.

$$\frac{5\left(1-\left(\frac{4}{5}\right)^k\right)}{1-\frac{4}{5}} > 24.95$$

$$25\left(1-\left(\frac{4}{5}\right)^k\right) > 24.95$$

$$1-\left(\frac{4}{5}\right)^k > \frac{499}{500}$$

$$\frac{1}{500} > \left(\frac{4}{5}\right)^k$$

$$\log\left(\frac{1}{500}\right) > \log\left(\left(\frac{4}{5}\right)^k\right)$$

$$\log(0.002) > k\log 0.8$$

We divide both sides by $\log 0.8$. Since $0.8 < 1$, $\log 0.8 < 0$ so we need to reverse the inequality sign.

$$k > \frac{\log(0.002)}{\log 0.8}$$

ii.

$k > \frac{\log(0.002)}{\log 0.8} \approx 27.9$. The smallest positive value of k is 28.

76)

We have $a = 20, r = \frac{7}{8}$. Let's substitute the formula for S_∞ and S_N into the inequality $S_\infty - S_N < 0.5$ and solve for N.

$$\frac{20}{1-\frac{7}{8}} - \frac{20\left(1-\left(\frac{7}{8}\right)^N\right)}{1-\frac{7}{8}} < 0.5$$

$$160 - 160\left(1-\left(\frac{7}{8}\right)^N\right) < 0.5$$

$$160 - 160 + 160\left(\frac{7}{8}\right)^N < 0.5$$

$$\left(\frac{7}{8}\right)^N < \frac{1}{320}$$

When taking log of base that is less than one, we need to reverse the inequality sign.

$$N > \log_{\frac{7}{8}}\frac{1}{320} \approx 43.198$$

Therefore, the smallest value of N is 44.

77)

i.

We have $a = 25000, r = 1.03$. We calculate the population at the end of year 2, $u_2 = ar = 25000(1.03) = 25750$

ii.

1.03

iii.

We are trying to solve $u_N > 40000$. Let's substitute the formula.

$$ar^{N-1} > 40000$$

$$25000(1.03)^{N-1} > 40000$$

$$1.03^{N-1} > 1.6$$

$$\log 1.03^{N-1} > \log 1.6$$

$$(N-1)\log 1.03 > \log 1.6$$

iv.

We continue from the previous part, dividing both sides by $\log 1.03$. Since $1.03 > 1$, $\log 1.03 > 0$.

$$N-1 > \frac{\log 1.6}{\log 1.03}$$
$$N > 1 + \frac{\log 1.6}{\log 1.03} \approx 16.9$$

Therefore $N = 17$

v.

Each year the amount given to charity is equal to u_n. So over 10 years this value will be s_n

$$s_n = \frac{a(1-r^{10})}{1-r} = \frac{25000(1-1.03^{10})}{1-1.03} = 286596.98 \approx £287000$$

3.2.6 With Trig

78)

This geometric progression has $a = 1, r = \frac{2\cos^2\theta}{1} = 2\cos^2\theta$.

To have an infinite sum, $-1 < r < 1$, but $2\cos^2\theta \geq 0$. So, we only need to make sure

$$2\cos^2\theta < 1$$
$$\cos^2\theta < \frac{1}{2}$$
$$-\frac{1}{\sqrt{2}} < \cos\theta < \frac{1}{\sqrt{2}}$$

Note: $2\cos^2\theta$ is always greater than or equal to zero so we could have also started by finding where $2\cos^2\theta = 1$ and used this to solve

We need to consider the interval $-\pi < \theta < \pi$:
We know $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, so in the interval $-\pi < \theta < \pi$, we can deduce that

$$-\frac{3\pi}{4} < \theta < \frac{\pi}{4} \quad or \quad \frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

i.

If θ is in the above region, we can calculate the sum to infinity using the formula $s_\infty = \frac{a}{1-r}$

$$s_\infty = \frac{a}{1-r} = \frac{1}{1-2\cos^2\theta} = -\frac{1}{2\cos^2\theta - 1} = -\frac{1}{\cos 2\theta} = -\sec 2\theta$$

79)

$$12\cos\theta, 5 + 2\sin\theta, 6\tan\theta$$

i.
we can build an equation based on knowing the sum of the ratio of successive terms must be the same

$$\frac{5+2\sin\theta}{12\cos\theta} = \frac{6\tan\theta}{5+2\sin\theta}$$

Now we solve

$$(5+2\sin\theta)(5+2\sin\theta) = 12\cos\theta(6\tan\theta)$$
$$25 + 20\sin\theta + 4\sin^2\theta = 72\cos\theta\frac{\sin\theta}{\cos\theta}$$
$$4\sin^2\theta + 20\sin\theta + 25 = 72\sin\theta$$
$$4\sin^2\theta - 52\sin\theta + 25 = 0$$

ii.

let $y = \sin\theta$

$$4y^2 - 52y + 25 = 0$$
$$(2y-1)(2y-25) = 0$$
$$y = \frac{1}{2} y = 12.5$$
$$\sin\theta = \frac{1}{2}, \sin\theta \neq 12.5$$
$$\theta = 30,°, 150°, \ldots$$

Given that θ is obtuse

$$\theta = 150°$$

iii.

The sequence becomes

$$12\cos 150°, 5 + 2\sin 150°, 6\tan 150°$$

We can simplify each term

$$-6\sqrt{3}, 6, -2\sqrt{3}$$

$a = -6\sqrt{3}$

$r = \frac{6}{-6\sqrt{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$$S_\infty = \frac{a}{1-r} = \frac{-6\sqrt{3}}{1 - -\frac{\sqrt{3}}{3}}$$

Let's multiply ALL terms by 3 to kill the fraction

$$-\frac{18\sqrt{3}}{3+\sqrt{3}}$$

Now we rationalize

$$-\frac{18\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3+-\sqrt{3}}$$

$$= \frac{-54\sqrt{3}+54}{9-3}$$

$$= \frac{-54\sqrt{3}+54}{6}$$

$$= -9\sqrt{3}+9$$

$$9(1-\sqrt{3})$$

$$k = 9$$

3.2.7 With Sectors

80)

There are two ways to approach this. First, we let the smallest sector have angle θ, and let the circular plank have radius r, so the area is πr^2. We also recall that the area of a sector in terms of its angle, θ, measured in radians is $\frac{r^2\theta}{2}$.

Method 1	**Method 2**
Let A be the smallest sector $(= u_1)$ of the progression, and let the common difference be d. We have $u_{12} = 2u_1$.	Let A be the smallest sector $(= u_1)$ of the progression, and let the common difference be d.

$$A + 11d = 2A$$
$$d = \frac{A}{11}$$

Now, we have the sum of all 12 sectors, s_{12}, is equal to the area of the circle.

$$s_{12} = \pi r^2$$
$$\frac{12}{2}(2A + (12-1)d) = \pi r^2$$
$$12A + 66d = \pi r^2$$

Substitute in $d = \frac{A}{11}$

$$12A + 66\frac{A}{11} = \pi r^2$$
$$18A = \pi r^2$$
$$A = \frac{\pi r^2}{18}$$

The smallest plank has area $A = \frac{r^2\theta}{2}$

$$\frac{r^2\theta}{2} = \frac{\pi r^2}{18}$$
$$\theta = \frac{\pi}{9}$$

The smallest sector has area $\frac{r^2\theta}{2}$, we're told that the largest plank (area u_{12}) has double the area of the smallest plank, so it has area $r^2\theta$.

So, $A = \frac{r^2\theta}{2}$, $u_{12} = A + 11d = r^2\theta$, substituting the former into the latter.

$$\frac{r^2\theta}{2} + 11d = r^2\theta$$
$$d = \frac{r^2\theta}{22}$$

The sum of all 12 sectors is equal to the area of the circle.

$$s_{12} = \pi r^2$$
$$\frac{12}{2}(2A + (12-1)d) = \pi r^2$$
$$12A + 66d = \pi r^2$$

Substitute $d = \frac{r^2\theta}{22}$, $A = \frac{r^2\theta}{2}$

$$6r^2\theta + 3r^2\theta = \pi r^2$$
$$9r^2\theta = \pi r^2$$
$$9\theta = \pi$$
$$\theta = \frac{\pi}{9}$$

81)

Let a sector have angle θ, then it has area $\frac{r^2\theta}{2}$, where r is the radius of the circle.

If another sector (of the same radius) has angle 2θ, then it has area $\frac{r^2(2\theta)}{2} = r^2\theta$. Its area is double that of the smaller circle.

So, if a sector has angle that is twice the angle of another sector, then it has area that is twice of that sector. We can conclude that this question is exactly the same as the previous question. The size of the angle of the smallest sector is $\frac{\pi}{9}$.

82)

OA= 1

By trigonometry, we have $\cos\theta = \frac{adj}{hyp} = \frac{OB}{OA}$

We also have

$\cos\theta = \frac{adj}{hyp} = \frac{OB_1}{OA}$

$\cos\theta = \frac{OB_1}{1}$

$OB_1 = \cos\theta$

We can generalize this to $OB_{n+1} = \cos\theta\,(OA_n)$, for all positive integers n.

Furthermore, since they are both radii of a sector, we have $OA_n = OB_n$, so $OB_{n+1} = \cos\theta\,(OB_n)$. This can be equivalently written as $OB_n = \cos^{n-1}\theta\,(OB_1) = \cos^{n-1}\theta\cos\theta = \cos^n\theta$.

Using the formula of arc length $r\theta$, each arc A_iB_i has length $(OB_i)\theta$.

The sum of arc lengths can now be written as the series

$$(OB)\theta + (OB_1)\theta + (OB_2)\theta + (OB_3)\theta + \cdots$$

We can apply our knowledge $OB_n = \cos^n\theta$, also we use the fact that OB is the radius, which is 1.

$$\theta + \theta\cos\theta + \theta\cos^2\theta + \theta\cos^3\theta + \cdots$$

This is a geometric series with $a = 1, r = \theta$. If it is the case that $-1 < \cos\theta < 1$, then we can substitute this into the formula for sum to infinity.

$$\frac{a}{1-r} = \frac{\theta}{1-\cos\theta}$$

4 Diamond

4.1 Using Formulae

4.1.1 Simultaneous Equations

83)

The first sequence has $a = 15, d = 19 - 15 = 4$. The second sequence has $a' = 420, d' = 415 - 420 = -5$. Let's equate the arithmetic series formula and solve for n.

$$\frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(2a' + (n-1)d')$$

Substitute our values for a, d, a', d'.

$$\frac{n}{2}(30 + 4(n-1) = \frac{n}{2}(840 - 5(n-1))$$

We can divide both sides by $\frac{n}{2}$, $n = 0$ is a fair assumption to make here.

$$30 + 4(n-1) = 840 - 5(n-1)$$
$$30 + 4n - 4 = 840 - 5n + 5$$
$$9n = 819$$
$$n = 91$$

84)

We have $a = 2k + 1, d = 4k + 4 - (2k + 1) = 4k + 4 - 2k - 1 = 2k + 3$.

Substitute this into the formula $u_n = a + (n-1)d$

$$u_{50} = 2k + 1 + (50 - 1)(2k + 3)$$
$$u_{50} = 2k + 1 + 49(2k + 3)$$
$$u_{50} = 2k + 1 + 98k + 147$$
$$u_{50} = 100k + 148$$

85)

We have $a = b^2 - 13, r = \frac{1}{b}$. Let's substitute these into the sum to infinity formula.

$$-6 = \frac{a}{1 - r}$$
$$-6 = \frac{b^2 - 13}{1 - \frac{1}{b}}$$
$$-6 + \frac{6}{b} = b^2 - 13$$
$$-6b + 6 = b^3 - 13b$$
$$b^3 - 7b - 6 = 0$$

Trial and error gives us $b = -1$ as a solution. We now need to factorise the cubic. Set

$$b^3 - 7b - 6 \equiv (b+1)(Ab^2 + Bb + C)$$
$$b^3 + 0b^2 - 7b - 6 \equiv Ab^3 + (A+B)b^2 + (B+C)b + C$$
$$A = 1$$
$$A + B = 0 \Rightarrow B = -1$$
$$C = -6$$

We have

$$b^3 - 7b - 6 = (b+1)(b^2 - b - 6) = 0$$
$$(b+1)(b-3)(b+2) = 0$$

So, the solutions are $b = -1, 3 \text{ } or - 2$.

But the sum to infinity formula requires that $-1 < b < 1$.

So, only $b = 3 \text{ } or - 2$ are valid. The possible common ratios are $\frac{1}{3}$ and $-\frac{1}{2}$.

86)

i.

We have $a = k, d = 2k - k = k$.

We want to find the number of terms in the series, so we want to find which term has value 100. We solve for $u_n = 100$, using the formula.

$$k + (n-1)k = 100$$
$$k + kn - k = 100$$
$$kn = 100$$
$$n = \frac{100}{k}$$

ii.

We substitute the number of terms $n = \frac{100}{k}$, the first term $a = k$, and the last term $l = 100$ into the formula $s_n = \frac{n}{2}(a + l)$

$$\frac{\frac{100}{k}}{2}(k + 100) = \frac{50}{k}(k + 100) = 50 + \frac{100}{k}$$

87)

We have $a = u_1 = 5t + 3, u_n = 17t + 11, d = 4$, we can solve for n in terms of t.

$$u_n = a + (n-1)d$$
$$17t + 11 = 5t + 3 + (n-1)(4)$$
$$17t + 11 = 5t + 3 + 4n - 4$$
$$4n = 12t + 12$$
$$n = 3t + 3$$

Now, we can use the formula $s_n = \frac{n}{2}(a + l)$ to work out the sum of the series.

$$s_n = \frac{3t+3}{2}(5t + 3 + 17t + 11)$$
$$s_n = \frac{3t+3}{2}(22t + 14)$$
$$s_n = (3t+3)(11t+7)$$

Now, if t is an odd number, $t = 2k + 1$ for some integer k.

$$s_n = (3(2k+1) + 3)(11(2k+3) + 7)$$
$$= (6k+6)(22k+40)$$
$$= \big(6(k+1)\big)\big(2(11k+20)\big)$$
$$= 12(k+1)(11k+20)$$

This is divisible by 12.

If t is an even number, $t = 2k$ for some integer k.

$$s_n = (3(2k)+3)(11(2k)+7)$$
$$= (6k+3)(22k+7)$$

Method 1. Using a parity argument	Method 2. Expand
Both brackets are odd numbers, as they are each an even number plus an odd number. The result is an odd number times an odd number, which is odd. But odd numbers are not divisible by 12. So this number is not divisible by 12.	$s_n = 132k^2 + 108k + 21$ $= 132k^2 + 108k + 12 + 9$ $= 12(11k^2 + 9k + 1) + 9$ This is a multiple of 12 plus 9, which cannot be a multiple of 12.

4.1.2 Arithmetic and Geometric Together in one question

88)

Hint: This is a mix of geometric and arithmetic think of 4,7,10 separately and then $2k, 4k, 8k$. Arithmetic is $3n-2$ and geometric is $2^{n-1}k$.

i.

We have $u_2 = S_2 - S_1, u_3 = S_3 - S_2, u_4 = S_4 - S_3$.

$$u_2 = 5 + 3k - 1 - k = 4 + 2k$$
$$u_3 = 12 + 7k - 5 - 3k = 7 + 4k$$
$$u_4 = 22 + 15k - 12 - 7k = 10 + 8k$$

i.

If we look at the sequence u_1, u_2, u_3, u_4.

$$1 + k, 4 + 2k, 7 + 4k, 10 + 8k$$

We can see that the constants form an arithmetic sequence with initial term $a = 1$, common difference $d = 3$. The terms containing k form a geometric sequence with initial term $b = k$, common ratio $r = 2$.

Then the arithmetic sequence has n^{th} term $1 + (n-1)(3) = 3n - 2$.
The geometric sequence has n^{th}term $k2^{n-1}$

So, the overall n^{th} term is $u_n = 3n - 2 + 2^{n-1}k$.

89)

We let the arithmetic sequence have initial term a, common difference d, and we let the geometric sequence have initial term b, common ratio r.

We can solve for a, d using our knowledge of the sums.

$$24 = s_3 = \frac{3}{2}(2a + (3-1)d)$$
$$24 = 3a + 3d$$

$$55 = s_5 = \frac{5}{2}(2a + (5-1)d$$
$$55 = 5a + 10d$$

Solving the above simultaneously gives us $a = 5, d = 3$.

So, the 3rd, 14th and 58th terms of the arithmetic sequence are

$$5 + 2(3), 5 + 13(3), 5 + 57(3)$$
$$11, 44, 176$$

We have $a = 11, r = \frac{44}{11} = 4$. Let's plug these values into the formula to find the sum of first 5 terms.

$$\frac{11(4^5 - 1)}{4 - 1} = \frac{11(1023)}{3} = 3751$$

90)

Hint: let the three numbers be $x, x + r, x + 2r$ and then they become $x - 1, x + r - 2, x + 2r$

We let the three numbers be $x, x+r, x+2r$, since they are in an arithmetic sequence. They sum to 24, so

$$x + x + r + x + 2r = 24$$
$$3x + 3r = 24$$
$$x + r = 8$$
$$x = 8 - r$$

We also have that $x-1, x+r-2, x+2r$ is in a geometric sequence.

$$\frac{x+r-2}{x-1} = \frac{x+2r}{x+r-2}$$
$$(x+r-2)^2 = (x+2r)(x-1)$$
$$x^2 + r^2 + 4 + 2rx - 4x - 4r = x^2 + 2rx - x - 2r$$
$$r^2 + 4 - 4x - 4r = -x - 2r$$
$$r^2 - 2r + 4 - 3x = 0$$

We can substitute $x = 8 - r$ into the quadratic.

$$r^2 - 2r + 4 - 3(8-r) = 0$$
$$r^2 + r - 20 = 0$$
$$(r+5)(r-4) = 0$$
$$r = -5 \text{ or } 4$$

We substitute the values of r back into $x = 8 - r$. We get

$$x = 13, r = -5 \text{ or } x = 4, r = 4$$

Substitute these back into the sequence

$$13, 8, 3 \text{ or } 4, 8, 12$$

91)

We let the common difference of the arithmetic sequence be d, let the common ratio of the geometric sequence be r.

We have $g_3 = a_2$

$$(1+\sqrt{5})r^2 = 1 + \sqrt{5} + d \quad ①$$

We also have $g_4 + a_3 = 0$

$$(1+\sqrt{5})r^3 + 1 + \sqrt{5} + 2d = 0 \quad ②$$

We can solve these two equations simultaneously. Rearrange ① to get

$$d = (1+\sqrt{5})r^2 - 1 - \sqrt{5}$$

Substitute this into ②

$$(1+\sqrt{5})r^3 + 1 + \sqrt{5} + 2\left((1+\sqrt{5})r^2 - 1 - \sqrt{5}\right) = 0$$
$$(1+\sqrt{5})r^3 + 1 + \sqrt{5} + 2(1+\sqrt{5})r^2 - 2 - 2\sqrt{5} = 0$$
$$(1+\sqrt{5})r^3 + (2+2\sqrt{5})r^2 - 1 - \sqrt{5} = 0$$

Divide both sides by $1+\sqrt{5}$

$$r^3 + 2r^2 - 1 = 0$$

Trial and error gives us $r = -1$ as a solution. We now need to factorise the cubic. Set

$$r^3 + 2r^2 - 1 \equiv (r+1)(Ar^2 + Br + C)$$
$$r^3 + 2r^2 + 0r - 1 \equiv Ar^3 + (A+B)r^2 + (B+C)r + C$$
$$A = 1$$
$$A + B = 2 \Rightarrow B = 1$$
$$C = -1$$

We have

$$r^3 + 2r^2 - 1 = (r+1)(r^2 + r - r)$$

Using the quadratic formula to solve the remaining two roots, we have $r = -1, \frac{-1-\sqrt{5}}{2}$ or $\frac{-1+\sqrt{5}}{2}$.

Since the geometric sequence has a sum to infinity, $-1 < r < 1$. So $r = \frac{-1+\sqrt{5}}{2}$.

Plugging into the sum to infinity formula.

$$\frac{a}{1-r}=\frac{1+\sqrt{5}}{1-\frac{-1+\sqrt{5}}{2}}=\frac{1+\sqrt{5}}{\frac{3-\sqrt{5}}{2}}=\frac{2+2\sqrt{5}}{3-\sqrt{5}}=\frac{(2+2\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{16+8\sqrt{5}}{9-5}=4+2\sqrt{5}$$

92)

Let the arithmetic sequence have first term a, common difference d. Let the geometric sequence have the first term b, common ratio r. We have the common difference is four time the first term of the geometric sequence.

$$d=4b$$

The common ratio is twice the first term of the arithmetic sequence.

$$r=2a$$

If we write out the formula for the first two terms of the sequence

$$\frac{3}{8}=a+b \text{ ①}$$
$$\frac{13}{16}=a+d+br$$

Substitute d, r.

$$\frac{13}{16}=a+4b+2ab \text{ ②}$$

We can rearrange ① to $a=\frac{3}{8}-b$, and substitute into ②.

$$\frac{13}{16}=\frac{3}{8}-b+4b+2\left(\frac{3}{8}-b\right)b$$
$$\frac{13}{16}=\frac{3}{8}-b+4b+\frac{3}{4}b-2b^2$$
$$2b^2-\frac{15}{4}b+\frac{7}{16}=0$$
$$32b^2-60b+7=0$$
$$b=\frac{1}{8}\ or\ \frac{7}{4}$$

4.2 With Binomial Expansion

93)

Using binomial expansion, we have

$$(1+kx)^n=1+nkx+\frac{n(n-1)}{2}k^2x^2+\cdots+\frac{n(n-1)(n-2)(n-3)}{24}k^4x^4+\cdots$$

We have $nk, \frac{n(n-1)}{2}k^2, \frac{n(n-1)(n-2)(n-3)}{24!}k^4$ are consecutive terms of a geometric sequence. They must have the same common ratio.

$$\frac{\frac{n(n-1)}{2}k^2}{nk}=\frac{\frac{n(n-1)(n-2)(n-3)}{24}k^4}{\frac{n(n-1)}{2}k^2}$$

Let's cancel some things out.

$$\frac{(n-1)k}{2}=\frac{(n-2)(n-3)k^2}{12}$$

Divide both sides by $k>0$, and multiplying out the denominators.

$$12(n-1)=2(n-2)(n-3)k$$
$$k=\frac{12(n-1)}{2(n-2)(n-3)}=\frac{6(n-1)}{(n-2)(n-3)}$$

4.3 With Logs

94)

We have $a = \frac{1}{\log_2 x}$,

$$d = \frac{1}{\log_8 x} - \frac{1}{\log_2 x}$$

We can use the formula $\log_b x = \frac{\log_c x}{\log_c b}$ to change the log to base 2.

$$d = \frac{1}{\frac{\log_2 x}{\log_2 8}} - \frac{1}{\log_2 x}$$
$$= \frac{\log_2 8}{\log_2 x} - \frac{1}{\log_2 x}$$
$$= \frac{3}{\log_2 x} - \frac{1}{\log_2 x}$$
$$= \frac{2}{\log_2 x}$$

We plug these values into $s_{20} = 100$, and solve for x.

$$\frac{20}{2}\left(2\left(\frac{1}{\log_2 x}\right) + (20-1)\left(\frac{2}{\log_2 x}\right)\right) = 100$$
$$\frac{2}{\log_2 x} + \frac{38}{\log_2 x} = 10$$
$$\frac{40}{\log_2 x} = 10$$
$$4 = \log_2 x$$
$$x = 16$$